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Mode III stress intensity factors for edge-cracked circular shafts, bonded wedges, bonded half planes and DCB's

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Abstract

Antiplane shear deformation of several edge-cracked geometries is considered. Analytical expressions are derived for the mode III stress intensity factor (SIF) of circular shafts with edge cracks, bonded half planes containing an interfacial edge crack, bonded wedges with an interfacial edge crack and also DCB's. The results are extracted for simple isotropic materials as well as anisotropic materials and also bonded dissimilar materials and it is shown that the same expressions are obtained for the SIF under the same geometries but with different above-mentioned material properties. Different boundary conditions are assumed and the SIF relations are derived in each case. As the special cases, the SIF's of the two bonded quarter planes containing an edge crack at the interface and infinite strip with a semi-infinite edge crack are extracted which coincide with the results cited in the literature.

1. Introduction

Analytical expressions for the stress intensity factors (SIF) of different geometries and various loadings are important in fracture mechanics. In the area of mode III problems, a number of contributions are related to the problem of finite or semi-infinite cracks in an infinite medium (Suo, 1989; Shiue et al., 1989; Choi et al., 1994; Lee and Earmme, 2000; Shahani and Adibnazari, 2000; Shahani, 2001). However, the interaction of finite boundaries on the cracks affects the severity of the induced stresses near the crack tip. Also, edge cracks vastly occur in composite laminates and bonded structures. Hence, edge delamination or edge debonding between the laminas or dissimilar components has appeared to become the main failure mode of these materials.

Most interfacial edge crack problems analyzed in the literature to date considered cracks between two bonded quarter planes. In the present paper, analytical expressions are derived for the stress intensity factors in different geometries including isotropic and anisotropic materials and dissimilar bi-materials. In all of the problems concentrated antiplane tractions are assumed to act which allows the solutions to be used as the Green's function for obtaining the SIF of any general distribution of tractions. At first, mode III

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SIF's are derived for a circular shaft containing an edge crack, under different boundary conditions on the crack faces and circular boundary of the shaft, made of isotropic as well as anisotropic materials. At the next part of the paper, first a closed form solution is extracted for the stress distribution in a bi-material infinite wedge, which has the advantage that shows geometric as well as load singularities explicitly, while the latter one is not apparent from the results previously published by the author. Then, the SIF in the problem of bonded half planes with an edge crack at the interface is obtained. Next, the SIF is extracted for a double cantilever beam (DCB) using the conformal mapping technique. Finally, the SIF is derived for bonded wedges containing an interfacial edge crack. It is shown that in all cases, the obtained results for the SIF's are independent of material property including isotropy, anisotropy and dissimilarity of the bonded materials. In the special cases, the results coincide with the published results in the literature.

2. Stress intensity factor in a circular shaft containing an edge crack

Kargarnovin et al. (1997) extracted explicit solutions for the stress distribution in an isotropic finite wedge under different boundary conditions. In the case when the edges of the wedge are subjected to concentrated antiplane tractions with equal distances from the apex ($h_1 = h_2 = h$) and the arc segments of the wedge is traction-free, letting the apex angle of the wedge equal to 2π, causes the wedge to resemble a circular shaft with an edge crack under antiplane shear stresses on the crack faces (Fig. 1).

In this case, the stress component $\tau_{\theta z}$ in the $r \leqslant h$ region reduces to

$$\tau_{\theta z}(r, \theta) = \frac{P}{h\pi} \sum_{k=0}^{\infty} \left(\frac{r}{h}\right)^{\frac{2k-1}{2}} \left[1 + \left(\frac{h}{a}\right)^{2k+1}\right] \sin\left(\frac{(2k+1)}{2}\theta\right); \quad r \leqslant h \tag{1}$$

Using the definition

$$K_{\mathrm{III}} = \lim_{r \to 0} \sqrt{2\pi r}\,\tau_{\theta z}(r, \pi) \tag{2}$$

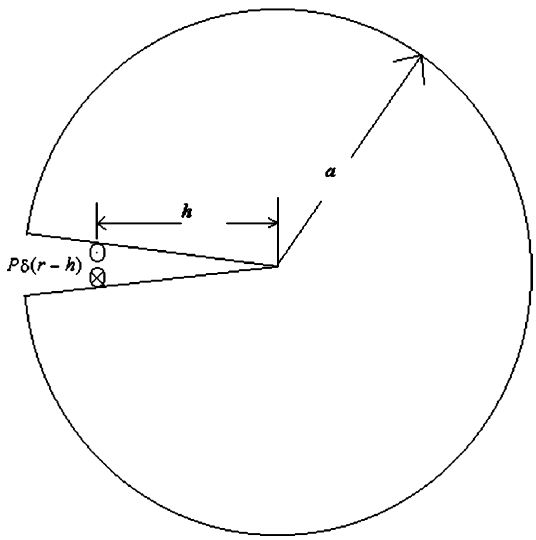

Fig. 1. Schematic view of a circular shaft containing an edge crack.

The stress intensity factor becomes

$$K_{\mathrm{III}} = \frac{\sqrt{2}P}{\sqrt{\pi h}}\left[1 + \frac{h}{a}\right] \tag{3}$$

If K_{III} is defined as

$$K_{\mathrm{III}} = \beta \frac{P}{\sqrt{\pi a}} \tag{4}$$

It is seen that β varies from $2\sqrt{2}$, when the point load is applied at the end of the crack face ($h = a$), to infinity, when the concentrated load is applied at the crack tip. If the shaft of Fig. 1 is fixed on its circumference instead of traction-free condition, the corresponding equation given by Kargarnovin et al. (1997) reduces to

$$\tau_{\theta z}(r,\theta) = \frac{P}{h\pi}\sum_{k=0}^{\infty}\left(\frac{r}{h}\right)^{\frac{2k-1}{2}}\left[1 - \left(\frac{h}{a}\right)^{2k+1}\right]\sin\left(\frac{(2k+1)}{2}\theta\right); \quad r \leqslant h \tag{5}$$

Using Eq. (2), we have

$$K_{\mathrm{III}} = \frac{\sqrt{2}P}{\sqrt{\pi h}}\left[1 - \frac{h}{a}\right] \tag{6}$$

Referring to the definition in Eq. (4), It is seen that β in this case varies from zero at $h = a$ to infinity, when the point load is applied at the crack tip.

From the results obtained by Kargarnovin et al. (1997), it may be seen that the familiar square root singularity at the crack tip would not be encountered in all cases, for example in the case when one of the crack faces is subjected to traction and the other to displacement boundary conditions, a singularity of the order $\frac{3}{4}$ is observed at the crack tip and thus computing the stress intensity factor from Eq. (2) gives an infinity one.

In a recent paper, Shahani (1999) has derived explicit analytical expressions for the stress distribution in an anisotropic finite wedge, subjected to different boundary conditions. Applying $\alpha = 2\pi$ (the wedge apex angle) in the traction-traction case, and then computing the stress intensity factor with the aid of Eq. (2), it is observed that the result is released from the effect of material property of the anisotropic material and the same result as Eq. (3) is obtained, as expected.

If the circular shaft of Fig. 1 is subjected to concentrated antiplane tractions on its circumference, as shown in Fig. 2, the problem should be analyzed separately. For this purpose, we consider the finite wedge of Fig. 3, with radius a, subjected to concentrated antiplane traction on its arc segment and fixed on one of its edges. The other edge of the wedge is traction-free. The constitutive equations for isotropic materials undergoing antiplane deformation reduce to

$$\begin{aligned} \tau_{rz} &= \mu\frac{\partial W}{\partial r} \\ \tau_{\theta z} &= \frac{\mu}{r}\frac{\partial W}{\partial \theta} \end{aligned} \tag{7}$$

The equilibrium equation of the problem in terms of the only non-zero displacement component, W, should be written as

$$\frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2} = 0 \tag{8}$$

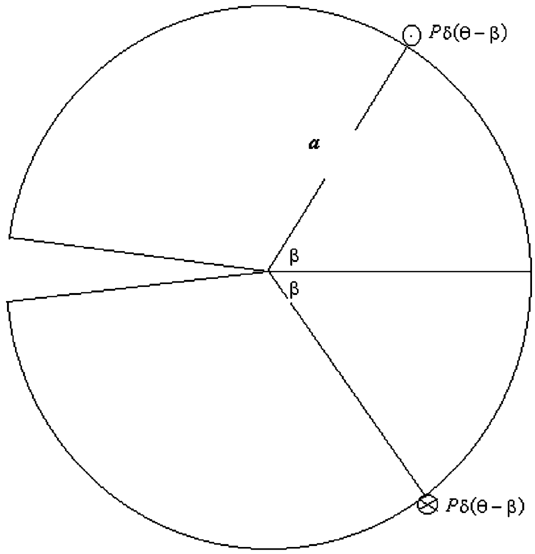

Fig. 2. Circular shaft with an edge crack subjected to antiplane concentrated tractions on its circumference.

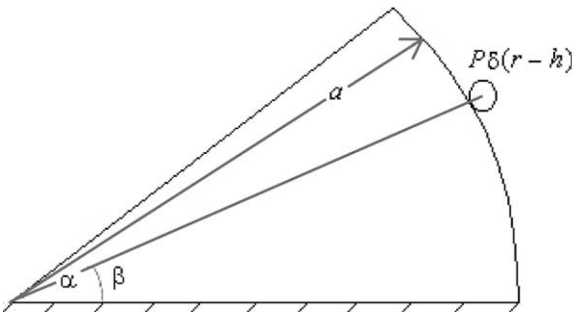

Fig. 3. Finite wedge with an antiplane concentrated traction on its circular segment.

where W is a function of in-plane coordinates (r, θ). The corresponding boundary conditions of the problem are as follows

$$\begin{aligned} &W(0,\theta) \text{ is a finite value} \\ &\tau_{rz}(a,\theta) = P\delta(\theta - \beta) \\ &W(r,0) = 0 \\ &\tau_{\theta z}(r,\alpha) = 0 \end{aligned} \tag{9}$$

Since the boundary conditions of the problem are homogeneous in the θ-direction, it can be solved by the method of separation of variables. The solution procedure is not given here, for the sake of brevity. But it may be shown that the displacement and stress components are obtained as follows

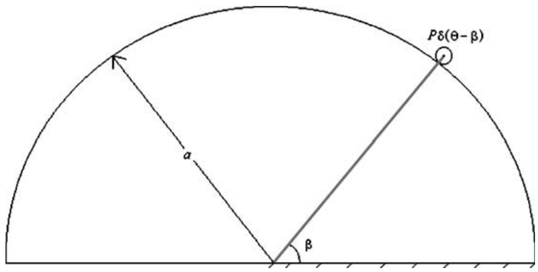

Fig. 4. Schematic view of a semi-circular shaft.

$$\begin{aligned} W(r,\theta) &= \frac{4Pa}{\mu\pi}\sum_{n=1}^{\infty}\frac{1}{2n+1}\left(\frac{r}{a}\right)^{\frac{(2n+1)\pi}{2\alpha}}\sin\left(\frac{(2n+1)\pi\beta}{2\alpha}\right)\sin\left(\frac{(2n+1)\pi\theta}{2\alpha}\right) \\ \tau_{rz}(r,\theta) &= \frac{2P}{\alpha}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^{\frac{(2n+1)\pi}{2\alpha}-1}\sin\left(\frac{(2n+1)\pi\beta}{2\alpha}\right)\sin\left(\frac{(2n+1)\pi\theta}{2\alpha}\right) \\ \tau_{\theta z}(r,\theta) &= \frac{2P}{\alpha}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^{\frac{(2n+1)\pi}{2\alpha}-1}\sin\left(\frac{(2n+1)\pi\beta}{2\alpha}\right)\cos\left(\frac{(2n+1)\pi\theta}{2\alpha}\right) \end{aligned} \tag{10}$$

Letting the apex angle equal to π ($\alpha = \pi$), we have a semi-circular shaft, as shown in Fig. 4, which is equivalent to the edge-cracked circular shaft, shown in Fig. 2, because of symmetry.

Thus, the stress intensity factor in the circular shaft of Fig. 2, may be calculated as

$$K_{\mathrm{III}} = 2\sqrt{2}P\sqrt{\frac{a}{\pi}}\sin\frac{\beta}{2} \tag{11}$$

3. The SIF of bonded half planes containing an edge crack at the interface

Bi-material wedges composed of two bonded isotropic wedges with dissimilar materials and infinite radius under concentrated antiplane shear tractions were recently analyzed by Shahani and Adibnazari (2000) and full-field solutions for the stress components were obtained. In the case of equal apex angles ($\theta_1 = \theta_2 = \alpha$), the following relation was given for the $r \leqslant h$ region, where h is the distance of the concentrated tractions to the wedge apex:

$$\tau^1_{\theta z}(r,\theta) = \frac{P}{h\alpha}\sum_{k=0}^{\infty}(-1)^k\left(\frac{r}{h}\right)^{\frac{(2k+1)\pi}{2\alpha}-1}\cos\left(\frac{(2k+1)\pi\theta}{2\alpha}\right); \quad r \leqslant h \tag{12}$$

It is observed that this relation is independent of material property and thus, the relation does not differ for similar material properties of the two bonded wedges. Hence, as explained about the equivalence of Figs. 2 and 4, the same result should be obtained from the corresponding equation of a finite simple wedge, with apex angle α, under traction–displacement boundary conditions, given by Kargarnovin et al. (1997), by letting $a \to \infty$. Defining $\rho = \frac{r}{h}$, we have $\rho < 1$ in the $r < h$ region and thus, Eq. (12) can be written as

$$\tau_{\theta z}(r,\theta) = \frac{P}{r\alpha}\rho^{\frac{\pi}{2\alpha}}\mathrm{e}^{\frac{\mathrm{i}\pi\theta}{2\alpha}}\mathrm{Re}\left\{\sum_{k=0}^{\infty}(-1)^k\rho^{\frac{k\pi}{\alpha}}\mathrm{e}^{\frac{\mathrm{i}k\pi\theta}{\alpha}}\right\} \tag{13}$$

where Re stands for the real part. Since the absolute value of the expression in the summation is less than unity, the following equality should be used

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}; \quad |x| < 1 \tag{14}$$

to write Eq. (13) in the form

$$\tau_{\theta z}(r,\theta) = \frac{P}{r\alpha}\rho^{\frac{\pi}{2\alpha}}\mathrm{Re}\left\{\frac{e^{\frac{i\pi\theta}{2\alpha}}}{1+\rho^{\frac{\pi}{\alpha}}e^{\frac{i\pi\theta}{\alpha}}}\right\} \tag{15}$$

Computing the real part and facilitating terms result in

$$\tau_{\theta z}(r,\theta) = \frac{P}{\alpha r^{1-\frac{\pi}{2\alpha}}h^{\frac{\pi}{2\alpha}}}\cos\left(\frac{\pi\theta}{2\alpha}\right)\frac{1+\frac{r}{h}\left(2\cos\left(\frac{\pi\theta}{\alpha}\right)-1\right)}{1+2\frac{r}{h}\cos\left(\frac{\pi\theta}{\alpha}\right)+\left(\frac{r}{h}\right)^2}; \quad r \leqslant h \tag{16}$$

This is a closed form solution for the stress component, $\tau_{\theta z}$, in the $r \leqslant h$ region. It has also two other advantages with respect to Eq. (12):

(i) The possible singularity, $r^{1-\frac{\pi}{2\alpha}}$, shows itself explicitly in the denominator of Eq. (16). The power of this term is the familiar order of stress singularity, which is in agreement with those published by Kargarnovin et al. (1997) and Shahani and Adibnazari (2000).

(ii) Applying the coordinates of the point of the application of the concentrated traction ($r = h$ and $\theta = \alpha$) in Eq. (12) gives no result other than zero, but applying $r = h$ and then $\theta = \alpha$ in Eq. (16) results in the infinity traction which corresponds to the load singularity arising from the concentrated nature of the applied traction.

Letting $\alpha = \pi$ in Eq. (16), the problem reduces to that of the two bonded half planes with an edge crack at the interface (Fig. 6(b)):

$$\tau_{\theta z}(r,\theta) = \frac{P}{\pi\sqrt{hr}}\cos\frac{\theta}{2}\frac{1+\frac{r}{h}(2\cos\theta-1)}{1+2\frac{r}{h}\cos\theta+\left(\frac{r}{h}\right)^2}; \quad r \leqslant h \tag{17}$$

It is seen that the familiar square root singularity is explicitly appeared in the above relation. Computing the stress intensity factor yields

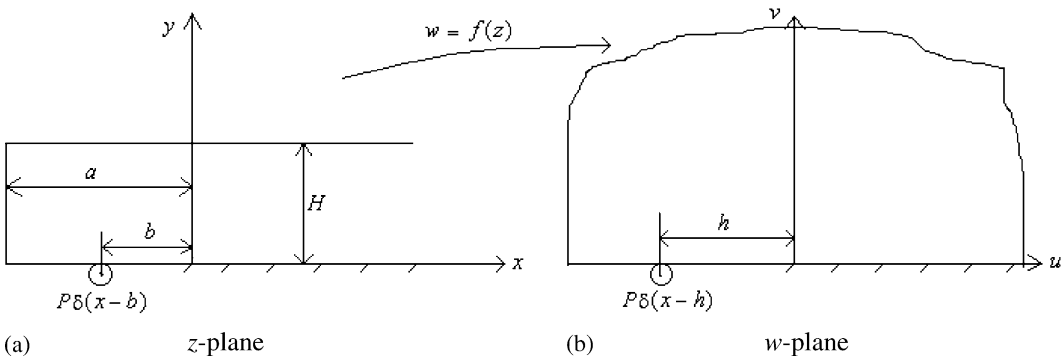

Fig. 5. (a) A semi-infinite strip under traction–displacement BC's on the boundary $y = 0$. (b) A half-space under the same type of BC's on the boundary $v = 0$.

$$K_{\text{III}} = \frac{\sqrt{2}P}{\sqrt{\pi h}} \tag{18}$$

The same result could also be obtained from Eqs. (3) and (6) by letting $a \to \infty$, but the above relations are extracted for concluding the results mentioned following Eq. (16) and also, for further applications in the next part of the paper.

4. The SIF in a DCB subjected to antiplane shear loading

The mode III stress intensity factor in a DCB can be derived from that of an elastic space containing an edge crack. The conformal mapping

$$w = f(z) = \frac{\cosh\left[\frac{\pi(z+a)}{H}\right]}{\cosh\left(\frac{\pi a}{H}\right)} - 1 \tag{19}$$

will map the strip of Fig. 5(a) onto the half-plane of Fig. 5(b). As previously explained, because of symmetry, these are equivalent to Fig. 6(a) and (b), respectively. It was seen in the previous section that the $\tau_{\theta z}$-stress component in the problem of Figs. 5(b) and 6(b) are of the form of Eq. (17). On the other hand, the $\tau_{\theta z}$-component can be written as the real part of a complex analytic function, in the w-plane

$$\tau_{\theta z} = \text{Re}[\Omega'(w)] \tag{20}$$

Now, using Eq. (15) with $\alpha = \pi$ and Eq. (20) and keeping in mind that $w = r\mathrm{e}^{\mathrm{i}\theta}$, the complex function $\Omega'(w)$ can be computed

$$\Omega'(w) = \frac{P\mathrm{e}^{\mathrm{i}\theta}}{\pi\sqrt{h}w^{\frac{1}{2}}}\frac{h}{h+w} \tag{21}$$

The complex function $\Omega'(w)$ can be expressed in terms of the complex variable z, in the z-plane, using the following relation:

$$\Omega'(z) = \Omega'(w) \cdot w'(z) \tag{22}$$

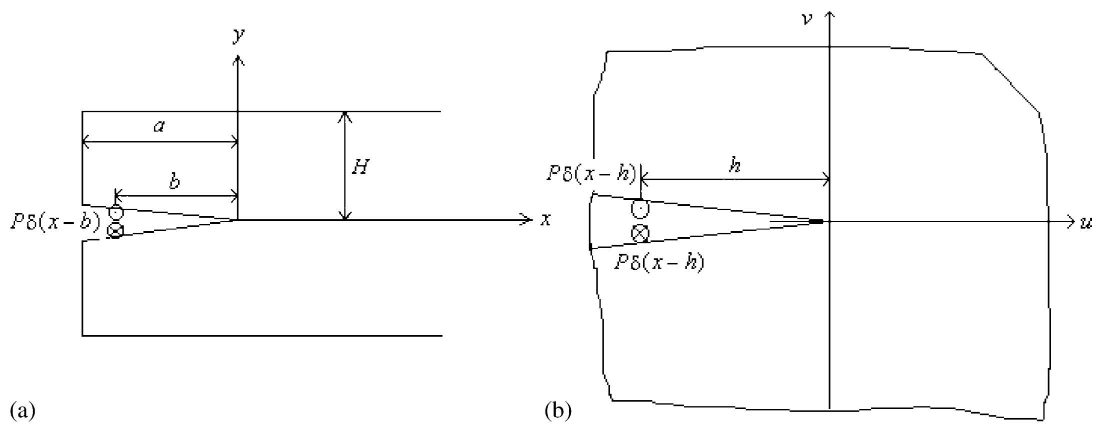

Fig. 6. (a) A DCB subjected to antiplane shear loading. (b) An edge-cracked space subjected to antiplane shear tractions on the crack faces.

The SIF in the problem of Figs. 5(a) or 6(a) can be derived using the following relation (Hellan, 1985),

$$K_{\mathrm{III}} = \lim_{z\to 0} \sqrt{2\pi z}\,\Omega'(z) \tag{23}$$

Noting that

$$w'(z) = \frac{\pi}{H} \frac{\sinh\left[\frac{\pi(z+a)}{H}\right]}{\cosh\left(\frac{\pi a}{H}\right)} \tag{24}$$

the SIF of the DCB under consideration can be derived as

$$K_{\mathrm{III}} = \frac{\sqrt{2}P}{\sqrt{hH}} \tanh^{\frac{1}{2}}\left(\frac{\pi a}{H}\right) \tag{25}$$

where h is the distance of the concentrated load from the crack tip in the w-plane and thus, should be expressed in terms of that in the z-plane using the transformation considered:

$$h = \frac{\cosh\left(\frac{\pi a}{H}\right) - \cosh\left[\frac{\pi(a-b)}{H}\right]}{\cosh\left(\frac{\pi a}{H}\right)} \tag{26}$$

Substituting this into Eq. (25), results in

$$K_{\mathrm{III}} = \frac{\sqrt{2}P}{\sqrt{H}} \frac{\sqrt{\sinh\left(\frac{\pi a}{H}\right)}}{\sqrt{\cosh\left(\frac{\pi a}{H}\right) - \cosh\left[\frac{\pi(a-b)}{H}\right]}} \tag{27}$$

It is observed that when the concentrated traction coincides with the crack tip ($b = 0$), the SIF approaches infinity. Furthermore, by letting $H \to \infty$, the problem reduces to that of two bonded elastic quarter planes with an interfacial edge crack:

$$K_{\mathrm{III}} = \frac{2P}{\sqrt{\pi b}} \sqrt{\frac{a}{2a-b}} \tag{28}$$

which is in agreement with the published results by Choi et al. (1994) and Lee and Earmme (2000). Meanwhile, letting $a \to \infty$ in Eq. (28), the problem resembles the infinite space with an edge crack of Fig. 6(b) and we obtain

$$K_{\mathrm{III}} = \frac{\sqrt{2}P}{\sqrt{\pi b}} \tag{29}$$

which coincides with Eq. (18). In addition, letting $a \to \infty$ directly in Eq. (27), the SIF for a semi-infinite crack in an infinite strip should be obtained

$$K_{\mathrm{III}} = \frac{P\sqrt{\frac{2}{H}}}{\sqrt{1 - \mathrm{e}^{\left(-\frac{\pi b}{H}\right)}}} \tag{30}$$

which coincides with the result in the literature (Sih, 1973) except for a constant $\sqrt{\pi}$ due to the difference in the SIF definition.

Now, considering that the concentrated tractions are applied at the edges of DCB, i.e., $z = -a + \mathrm{i}c$, the transformed distance of the load to the crack tip in the w-plane will be

$$h = \frac{\cosh\left(\frac{\pi a}{H}\right) - \cos\left(\frac{\pi c}{H}\right)}{\cosh\left(\frac{\pi a}{H}\right)} \tag{31}$$

Substituting this relation into Eq. (25), the SIF of the DCB in this case will be

$$K_{\mathrm{III}} = \frac{\sqrt{2}P}{\sqrt{H}} \frac{\sqrt{\sinh\left(\frac{\pi a}{H}\right)}}{\sqrt{\cosh\left(\frac{\pi a}{H}\right) - \cos\left(\frac{\pi c}{H}\right)}} \tag{32}$$

In the special case when $H \to \infty$, the problem reduces to that of two bonded quarter planes with an interfacial edge crack subjected to antiplane concentrated tractions on the free edges of the quarter planes and computing the limiting value of Eq. (32) yields

$$K_{\mathrm{III}} = \frac{2P}{\sqrt{\pi}} \sqrt{\frac{a}{a^2 + c^2}} \tag{33}$$

5. The SIF in bonded wedges containing an interfacial edge crack

Shahani (2001) proved that when the first tip of a finite crack, lying at the interface of two bonded wedges with equal apex angles α (Fig. 7), coincides with the apex, the singularity at this tip vanishes and the following result could be obtained:

$$f(r) = -\frac{P\sqrt{b^\gamma + h^\gamma}}{\mu_e \alpha} \frac{r^{\gamma-1}(b^\gamma - r^\gamma)^{-\frac{1}{2}}}{r^\gamma + h^\gamma}; \quad \gamma = \frac{\pi}{\alpha} \tag{34}$$

where $f(r)$ is the density function of the screw dislocations, b indicates the crack length in this case, h is the distance of the concentrated tractions to the wedge apex and μ_e is the equivalent shear modulus of the bimaterial wedge and is defined as $\mu_e = \frac{\mu_1 \mu_2}{\mu_1 + \mu_2}$. It is now possible to calculate the SIF for the edge-cracked composite wedge of Fig. 7, using the following definition:

$$K_{\mathrm{III}} = -\lim_{r \to b} \mu_e \sqrt{2\pi(b - r)} f(r) \tag{35}$$

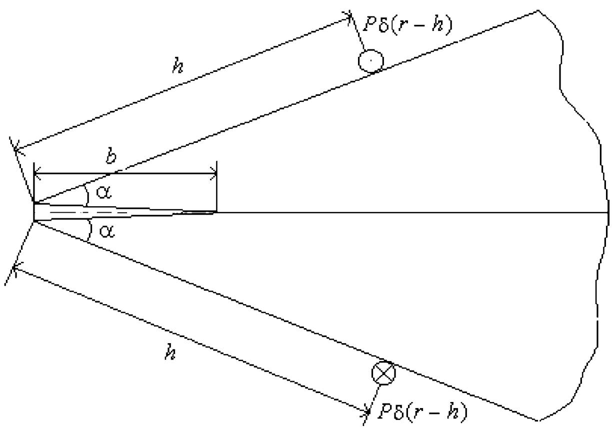

Fig. 7. Bonded wedges with an interfacial edge crack.

which gives after some mathematical procedures:

$$K_{\mathrm{III}} = P\sqrt{\frac{2}{\alpha}}\sqrt{\frac{b^{\gamma-1}}{b^{\gamma}+h^{\gamma}}}; \quad \gamma = \frac{\pi}{\alpha} \tag{36}$$

In the special case of bonded quarter planes with an interfacial edge crack $\left(\alpha = \frac{\pi}{2}\right)$, Eq. (36) gives

$$K_{\mathrm{III}} = \frac{2P}{\sqrt{\pi}}\sqrt{\frac{b}{b^2+h^2}} \tag{37}$$

Keeping in mind the difference between the notations of the crack length and the distance of concentrated tractions from the apex, this is exactly the same as Eq. (33).

Another special case can be deduced by letting $\alpha = \pi$, which corresponds to the problem of bonded dissimilar half planes with interfacial edge crack where the concentrated tractions are applied at a distance $b + h$ from the crack tip:

$$K_{\mathrm{III}} = \frac{\sqrt{2}P}{\sqrt{\pi(b+h)}} \tag{38}$$

which is coincident with the results obtained in Eqs. (18) and (29).

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